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A COMMON FIXED POINT THEOREM FOR A PAIR OF SELF MAPPINGS SATISFYING A GENERAL CONTRACTIVE CONDITION OF EXPONENTIAL TYPE

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Abstract

In this paper, we prove a common fixed point theorem for a pair of self mappings satisfying a general contractive condition of exponential type in complete metric spaces. Our results extend and improve the results of Sharma and Ojha [1].

1. Introduction

The first well known result on fixed points for a contractive mapping was Banach's fixed point theorems, published in 1922 (see [2, 3]). In general setting of complete metric space, Smart [4] presented the following result as well as [9, 10]:

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Theorem 1.1. *Let* (X, d) *be a complete metric space*, $c \in [0, 1)$ *and let* $T : X \to X$ *be a mapping such that for each* $x, y \in X$,

$$d(Tx, Ty) \leq cd(x, y).$$

Then T *has a unique fixed point* $z \in X$ *such that for each* $x \in X$, $\lim_{n\to\infty} T^n x = z$.

After this classical result, many theorems dealing with maps satisfying various types of contractive inequalities have been established (see [5-8], [12-16]). Sharma and Ojha [1] generalized the result of Banach's fixed point by introducing the following theorem:

Theorem 1.2. *Let* (X, d) *be a complete metric space*, $c \in [0, 1)$ *and let* $T : X \to X$ *be a map such that for every* $x, y \in X$,

$$e^{d(Tx, Ty)} \leq ce^{d(x, y)},$$

where $e : R_+ \to R_+$ *is a Lebesgue-integrable map which is summable*, *positive and such that* $e^{\varepsilon} > 0$ *for each* $\varepsilon > 0$. *Then* T *has a unique fixed point* $z \in X$ *and for each*, $\lim_{n\to\infty} T^n x = z$.

The main object of this paper is to obtain some results for a pair of self maps satisfying a general contractive condition of exponential type for four mappings.

Definition 1.3 [9]**.** Let f and g be two mappings from a metric space (X, d) into itself. Then maps f and g are called:

(i) *weakly compatible* if they commute at their coincidence point, i.e., $fx = gx$ for some $x \in X$ implies $fgx = gfx$.

(ii) *compatible* if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some t in X.

(iii) *commuting* if

$$fgx = gfx \quad \forall x \in X.$$

Definition 1.4 [9]**.** Let f and g be two mappings on a set X. If $fx = gx$ for some x in X, then x is called *coincidence point* of f and g:

2. Main Results

We begin with the following theorem:

Theorem 2.1. *Let (X, d) be a complete metric space. Let a_i $(i = 1, 2, 3, 4, 5)$ be nonnegative real numbers satisfying $\sum_{i=1}^{5} a_i < 1$, T_1, T_2, f and g are four self maps of X satisfying the following conditions*:

(i) $T_1(X) \subseteq f(X)$ *and* $T_2(X) \subseteq g(X)$,

(ii) *the pairs* (T_2, f) *and* (T_1, g) *are weakly compatible*,

(iii)

$$e^{d(T_1x, T_2y)} \leq a_1 e^{d(fx, gy)} + a_2 e^{d(fx, T_1x)} + a_3 e^{d(gy, T_2y)} + a_4 e^{d(fx, T_2y)} + a_5 e^{d(gy, T_1x)}, \qquad (1)$$

where $e : R_+ \to R_+$ is a Lebesgue-integrable map which is summable, non-negative and such that $e^{\varepsilon} > 0$ for each $\varepsilon > 0$. Then T_1, T_2, f and g have a unique common fixed point $z \in X$.

Proof. Suppose that x_0 is an arbitrary point of *X* and define the sequence $\{y_n\}$ in *X* such that

$$y_n = T_2x_n = gx_{n+1} \text{ and } y_{n+1} = T_1x_{n+1} = fx_{n+2}. \tag{2}$$

By interchanging *x* with *y*, T_1 with T_2 and *f* with *g*, we obtain

$$e^{d(T_2y, T_1x)} \leq a_1e^{d(gy, fx)} + a_2e^{d(gy, T_2y)} + a_3e^{d(fx, T_1x)} + a_4e^{d(gy, T_1x)} + a_5e^{d(fx, T_2y)}. \tag{3}$$

Now from (1), (3) and using symmetric property, we have

$$e^{d(T_1x, T_2y)} \leq a_1e^{d(fx, gy)} + \left(\frac{a_2 + a_3}{2}\right)e^{d(fx, T_1x)} + \left(\frac{a_2 + a_3}{2}\right)e^{d(gy, T_2y)} + \left(\frac{a_4 + a_5}{2}\right)e^{d(fx, T_2y)} + \left(\frac{a_4 + a_5}{2}\right)e^{d(gy, T_1x)}. \tag{4}$$

Using (4), for even *n*, we obtain

$$\begin{aligned} e^{d(y_n, y_{n+1})} &= e^{d(T_1x_n, T_2x_{n+1})} \\ &\leq a_1e^{d(fx_n, gx_{n+1})} + \left(\frac{a_2 + a_3}{2}\right)e^{d(fx_n, T_1x_n)} \\ &+ \left(\frac{a_2 + a_3}{2}\right)e^{d(gx_{n+1}, T_2x_{n+1})} \\ &+ \left(\frac{a_4 + a_5}{2}\right)e^{d(fx_n, T_2x_{n+1})} + \left(\frac{a_4 + a_5}{2}\right)e^{d(gx_{n+1}, T_1x_n)}. \end{aligned}$$

From (2), we can write

$$e^{d(y_n, y_{n+1})} \leq a_1e^{d(y_{n-1}, y_n)} + \left(\frac{a_2 + a_3}{2}\right)e^{d(y_{n-1}, y_n)} + \left(\frac{a_2 + a_3}{2}\right)e^{d(y_n, y_{n+1})} + \left(\frac{a_4 + a_5}{2}\right)e^{d(y_{n-1}, y_{n+1})} + \left(\frac{a_4 + a_5}{2}\right)e^{d(y_n, y_n)}.$$

Again, applying (4), for even n, we obtain

$$e^{d(y_n, y_{n+1})} = e^{d(T_2 x_n, T_1 x_{n+1})}$$

$$\leq a_1 e^{d(fx_n, gx_{n+1})} + \left(\frac{a_2 + a_3}{2}\right) e^{d(fx_n, T_1 x_n)}$$

$$+ \left(\frac{a_2 + a_3}{2}\right) e^{d(gx_{n+1}, T_2 x_{n+1})}$$

$$+ \left(\frac{a_4 + a_5}{2}\right) e^{d(fx_n, T_2 x_{n+1})} + \left(\frac{a_4 + a_5}{2}\right) e^{d(gx_{n+1}, T_1 x_n)}.$$

By (2), we have

$$e^{d(y_n, y_{n+1})} \leq a_1 e^{d(y_{n-1}, y_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(y_{n-1}, y_n)}$$

$$+ \left(\frac{a_2 + a_3}{2}\right) e^{d(y_n, y_{n+1})} + \left(\frac{a_4 + a_5}{2}\right) e^{d(y_{n-1}, y_{n+1})}$$

$$+ \left(\frac{a_4 + a_5}{2}\right) e^{d(y_n, y_n)}.$$

The above two inequalities yields,

$$e^{d(y_n, y_{n+1})} \leq a_1 e^{d(y_{n-1}, y_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(y_{n-1}, y_n)}$$

$$+ \left(\frac{a_2 + a_3}{2}\right) e^{d(y_n, y_{n+1})} + \left(\frac{a_4 + a_5}{2}\right) e^{d(y_{n-1}, y_{n+1})}$$

$$+ \left(\frac{a_4 + a_5}{2}\right) e^{d(y_n, y_n)}$$

$$\leq a_1 e^{d(y_{n-1}, y_n)} + \left(\frac{a_2 + a_3}{2}\right) e^{d(y_{n-1}, y_n)}$$

$$+ \left(\frac{a_2 + a_3}{2}\right) e^{d(y_n, y_{n+1})} + \left(\frac{a_4 + a_5}{2}\right) e^{d(y_{n-1}, y_n)}$$

$$+ \left(\frac{a_4 + a_5}{2}\right) e^{d(y_n, y_{n+1})}.$$

It follows that

$$e^{d(y_n, y_{n+1})} \leq \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5} e^{d(y_{n-1}, y_n)}$$

$$= re^{d(y_{n-1}, y_n)}$$

$$\leq r^n e^{d(y_1, y_0)} \to 0 \text{ as } n \to \infty,$$

since $r < 1,$ owing to the assumption $\sum_{i=1}^{5} a_i < 1.$ Therefore

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \tag{5}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in X. Let $m > n$ where $m, n \in N$ without any loss of concepts, here two cases arises:

Case 1. We choose n and m to be odd and even respectively, then we have

$$e^{d(y_n, y_m)} = e^{d(T_1 x_n, T_2 x_m)}$$

$$\leq a_1 e^{d(fx_n, gx_m)} + a_2 e^{d(fx_n, T_1 x_n)} + a_3 e^{d(gx_m, T_2 x_m)}$$

$$+ a_4 e^{d(fx_n, T_2 x_m)} + a_5 e^{d(gx_m, T_1 x_n)}$$

$$\leq a_1 e^{d(y_{n-1}, y_{m-1})} + a_2 e^{d(y_{n-1}, y_n)} + a_3 e^{d(y_{m-1}, y_m)}$$

$$+ a_4 e^{d(y_{n-1}, y_m)} + a_5 e^{d(y_{m-1}, y_n)}.$$

Case 2. We choose n and m to be even and odd respectively, then we have

$$e^{d(y_n, y_m)} = e^{d(T_2 x_n, T_1 x_m)}$$

$$\leq a_1 e^{d(fx_n, gx_m)} + a_2 e^{d(fx_n, T_1 x_n)} + a_3 e^{d(gx_m, T_2 x_m)}$$

$$+ a_4 e^{d(fx_n, T_2 x_m)} + a_5 e^{d(gx_m, T_1 x_n)}$$

$$\leq a_1 e^{d(y_{n-1}, y_{m-1})} + a_2 e^{d(y_{n-1}, y_n)} + a_3 e^{d(y_{m-1}, y_m)}$$

$$+ a_4 e^{d(y_{n-1}, y_m)} + a_5 e^{d(y_{m-1}, y_n)}.$$

From the above two cases, we get

$$e^{d(y_n, y_m)} \leq a_1 e^{d(y_{n-1}, y_{m-1})} + a_2 e^{d(y_{n-1}, y_n)} + a_3 e^{d(y_{m-1}, y_m)}$$

$$+ a_4 e^{d(y_{n-1}, y_m)} + a_5 e^{d(y_{m-1}, y_n)}$$

$$\leq a_1 e^{d(y_{n-1}, y_n)} + a_1 e^{d(y_n, y_m)} + a_1 e^{d(y_{m-1}, y_m)}$$

$$+ a_2 e^{d(y_{n-1}, y_n)} + a_3 e^{d(y_{m-1}, y_m)}$$

$$+ a_4 e^{d(y_{n-1}, y_n)} + a_4 e^{d(y_n, y_m)} + a_5 e^{d(y_{m-1}, y_m)} + a_5 e^{d(y_m, y_n)}$$

$$\leq \left(\frac{a_1 + a_2 + a_4}{1 - a_1 - a_4 - a_5} \right) e^{d(y_{n-1}, y_n)} + \left(\frac{a_1 + a_2 + a_5}{1 - a_1 - a_4 - a_5} \right) e^{d(y_{m-1}, y_m)}$$

$$\leq r^{n-1} e^{d(y_1, y_0)} + r^{m-1} e^{d(y_1, y_0)} \to 0 \text{ as } n, m \to \infty.$$

Hence $\{y_n\}$ is a Cauchy sequence in a complete metric space X, so it is convergent in X. Let its limit be z, i.e., $\lim_{n \to \infty} y_n = z$. Hence

$$\lim_{n \to \infty} T_2 x_n = \lim_{n \to \infty} g x_{n+1} = \lim_{n \to \infty} T_1 x_{n+1} = \lim_{n \to \infty} f x_{n+2} = z. \tag{6}$$

Since $T_1(X) \subseteq f(X)$, there exists a point $u \in X$ such that $z = fu$. Then from (1), we get

$$e^{d(y_n, T_2 u)} = e^{d(T_1 x_{n+1}, T_2 u)} \leq a_1 e^{d(fx_{n+1}, gu)} + a_2 e^{d(fx_{n+1}, T_1 x_{n+1})}$$

$$+ a_3 e^{d(gu, T_2 u)} + a_4 e^{d(fx_{n+1}, T_2 u)} + a_5 e^{d(gu, T_1 x_{n+1})}.$$

Taking the limit as $n \to \infty$ and by (6), we have

$$e^{d(z,T_2u)} \leq a_1e^{d(z,gu)} + a_2e^{d(z,z)} + a_3e^{d(gu,T_2u)} + a_4e^{d(z,T_2u)} + a_5e^{d(gu,z)}$$

$$\leq a_1e^{d(z,gu)} + a_3e^{d(gu,z)} + a_3e^{d(z,T_2u)} + a_4e^{d(z,T_2u)} + a_5e^{d(gu,z)}$$

$$\leq \left(\frac{a_1 + a_3 + a_5}{1 - a_3 + a_5}\right)e^{d(z,gu)}, \tag{7}$$

the inequality (7) is valid if

$$e^{d(z,T_2u)} = 0 \text{ implies } z = T_2u,$$

so $z = T_2u = fu.$ Hence u is a coincidence point of f and T_2. Since the pair (f, T_2) is weakly compatible,

$$T_2fu = fT_2u \text{ implies } T_2z = fz. \tag{8}$$

Again, since $T_2(X) \subseteq g(X)$, there exists a point $v \in X$ such that $z = gv$. Then by (1) and applied the same above steps, we can find that $T_1v = z$, so $z = T_1v = gv$. Hence v is a coincidence point of g and T_1.

Also, the pair of maps T_1 and g are weakly compatible, i.e.,

$$gT_1v = T_1gv \text{ implies } gz = T_1z. \tag{9}$$

Now, we show that z is a fixed point of T_2, by using (1), we have

$$e^{d(z,T_2z)} = e^{d(T_1x_{n+1},T_2z)} \leq a_1e^{d(fx_{n+1},gz)} + a_2e^{d(fx_{n+1},T_1x_{n+1})}$$

$$+ a_3e^{d(gz,T_2z)} + a_4e^{d(fx_{n+1},T_2z)} + a_5e^{d(gz,T_1x_{n+1})}.$$

Taking the limit as $n \to \infty$, we get

$$e^{d(z,T_2z)} \leq a_1e^{d(z,gz)} + a_2e^{d(z,z)} + a_3e^{d(gz,T_2z)}$$

$$+ a_4e^{d(z,T_2z)} + a_5e^{d(gz,z)}$$

$$\leq a_1 e^{d(z, gz)} + a_3 e^{d(gz, z)} + a_3 e^{d(z, T_2 z)} + a_4 e^{d(z, T_2 z)} + a_5 e^{d(gz, z)}$$

$$\leq \left(\frac{a_1 + a_3 + a_5}{1 - a_3 + a_4}\right) e^{d(z, gz)} \leq r e^{d(z, gz)} < e^{d(z, gz)}, \tag{10}$$

the inequality (10) is valid if

$$e^{d(z, T_2 z)} = 0 \text{ implies } z = T_2 z.$$

Applying (8), it follows that

$$z = T_2 z = fz. \tag{11}$$

Also, by the same way we can show that z is a fixed point of T_1, hence $z = T_1 z$ from (9) we can write

$$z = T_1 z = gz. \tag{12}$$

From (12) and (11), we obtain that

$$z = T_2 z = fz = T_1 z = gz.$$

Therefore z is a common fixed point of T_1, T_2, f and g.

For uniqueness of z let if possible that z and w are common fixed points of T_1, T_2, f and g such that $(w \neq z)$, from (1), we have

$$\begin{aligned} e^{d(z, w)} = e^{d(T_1 z, T_2 w)} &\leq a_1 e^{d(fz, gw)} + a_2 e^{d(fz, T_1 z)} + a_3 e^{d(gw, T_2 w)} \\ &+ a_4 e^{d(fz, T_2 w)} + a_5 e^{d(gw, T_1 z)} \\ &\leq a_1 e^{d(z, w)} + a_2 e^{d(z, z)} + a_3 e^{d(w, w)} + a_4 e^{d(z, w)} + a_5 e^{d(w, z)} \\ &\leq (a_1 + a_4 + a_5) e^{d(z, w)} \\ &= r e^{d(z, w)} \\ &< e^{d(z, w)}, \end{aligned}$$

which is a contradiction (since $r < 1$), so $z = w$, i.e., z is a unique common fixed point of T_1, T_2, f and g. This completes the proof.

If we put $f = g$ in the above theorem, we get the following corollary:

Corollary 2.1. *Let* (X, d) *be a complete metric space and* a_i $(i = 1, 2, 3, 4, 5)$ *be nonnegative real numbers satisfying* $\sum_{i=1}^{5} a_i < 1,$ $T_1,$ T_2 *and f are self maps on X satisfying the following conditions*:

(i) $T_1(X) \subseteq f(X)$ *and* $T_2(X) \subseteq f(X),$

(ii) *the pairs* (T_2, f) *and* (T_1, f) *are weakly compatible*,

(iii)

$$e^{d(T_1x, T_2y)} \leq a_1 e^{d(fx, fy)} + a_2 e^{d(fx, T_1x)} + a_3 e^{d(fy, T_2y)} + a_4 e^{d(fx, T_2y)} + a_5 e^{d(fy, T_1x)}.$$

Then T_1, T_2 *and f have a unique common fixed point* $z \in X$.

Corollary 2.2. *Let* (X, d) *be a complete metric space and a*, *b*, *c be positive real numbers satisfying* $a + b + c < 1$, T_1, T_2, *f and g are four self maps of X satisfying the following conditions*:

(i) $T_1(X) \subseteq f(X)$ *and* $T_2(X) \subseteq g(X),$

(ii) *the pairs* (T_2, f) *and* (T_1, g) *are weakly compatible*,

(iii)

$$e^{d(T_1x, T_2y)} \leq ae^{d(fx, gy)} + be^{d(fx, T_1x)} + ce^{d(gy, T_2y)}.$$

Then T_1, T_2, *f and g have a unique common fixed point* $z \in X$.

Proof. The result follows immediately from Theorem 2.1, by taking $a_4 = a_5 = 0,$ $a_1 = a,$ $a_2 = b$ and $a_3 = c.$

Corollary 2.3. *Let* (X, d) *be a complete metric space and a*, *b*, *c be positive real numbers satisfying* $a + b + c < 1$, T_1, T_2, *f and g are four self maps on X satisfying the following conditions*:

(i) $T_1(X) \subseteq f(X)$ *and* $T_2(X) \subseteq g(X),$

(ii) *the pair* (T_2, f) *and* (T_1, g) *are weakly compatible*,

(iii)

$$e^{d(T_1x, T_2y)} \leq ae^{d(fx, gy)} + be^{d(fx, T_2y)} + ce^{d(gy, T_1x)}$$

Then T_1, T_2, f *and* g *have a unique common fixed point* $z \in X$.

Proof. Taking $a_2 = a_3 = 0, a_1 = a, a_4 = b$ and $a_5 = c$ in Theorem 2.1, we get the proof.

Remark. If we take $f = g = I$ (where I is the identity mapping), $T_1 = T_2 = T, a_2 = a_3 = a_4 = a_5 = 0$ and $a_1 = c, 0 \leq c < 1$ in Theorem 2.1, we obtain Theorem 1.2.

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